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LETTER TO THE EDITOR

Algebraic areas enclosed by 2D Brownian curves in random media

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Abstract. We study closed Brownian trajectories when the Brownian particle is subjected to a random potential. For a Poissonian δ repulsive potential, the enclosed algebraic area, A, is a Gaussian variable that scales like $t^{3/4}$ when t, the length of the curve, goes to infinity. This is intermediate between the situations where the particle is allowed to wander (i) everywhere on the plane ($A \sim t$) and (ii) only on a bounded domain ($A \sim t^{1/2}$). For the Lloyd model, we show that the probability distribution, $P(A)$, is the same as in the absence of disorder. This surprising result is related to some peculiarities of the Cauchy law.

In this letter, we will address the problem of the area, A, enclosed by the trajectory of a Brownian particle when this particle is submitted to a random potential. Each Brownian curve will be weighted by a factor $\exp(-\int_0^t V(\vec{r}(\tau))\,\mathrm{d}\tau)$ where, in a first step, we choose for V ($\lambda > 0$):

$$V(\vec{r}) = \lambda \sum_i \delta(\vec{r} - \vec{r}_i). \tag{1}$$

The locations $\vec{r}_i$ of the scattering centres are randomly distributed on the plane according to a Poisson's law with an average density ρ. Before computing the probability distribution, $P(A)$, averaged over the set of positions $\{\vec{r}_i\}$, we recall some results that will be useful for our work.

The study of the algebraic area, A, enclosed by a Brownian curve of length t traces back to the pioneering work of Levy [1]. Considering a Brownian particle allowed to wander everywhere on the plane, he got, for $P(A)$, the result:

$$P(A) = \frac{\pi}{2t} \frac{1}{\cosh^2\left(\frac{\pi A}{t}\right)} \tag{2}$$

where clearly A scales like t.

In a path integral formulation, we can write $P(A)$ as:

$$P(A) = N \int \mathrm{d}\vec{r} \int_{\vec{r}(0)=\vec{r}(t)=\vec{r}} \mathrm{D}\vec{r}\,(\tau) \delta\left(A - \tfrac{1}{2}\int_0^t r^2 \dot{\theta}\,\mathrm{d}\tau\right) \exp\left(-\int_0^t \tfrac{1}{2}\dot{\vec{r}}^2(\tau)\,\mathrm{d}\tau\right) \tag{3}$$

(N is a normalization constant).

Using the identity $\delta(x) = (1/2\pi)\int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}Bx}$, (3) becomes

$$P(A) = N' \int_{-\infty}^{+\infty} \mathrm{d}B \mathrm{e}^{\,\mathrm{i}BA}\, \mathrm{Tr}(\mathrm{e}^{-tH}) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}BA} \left(\frac{Z(B)}{Z(0)}\right) \tag{4}$$

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where H is the Landau Hamiltonian of a charged particle in a constant magnetic field:

$$H = \frac{1}{2}\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(-\mathrm{i}\partial_\theta - \frac{Br^2}{2}\right)^2\right). \tag{5}$$

Equation (2) is recovered by computing the Landau partion function, $Z(B)$, and performing the Fourier transform according to (4).

The work of Levy has been extended in various directions [2, 3]. For instance, if we consider a Brownian particle allowed to wander only on a bounded domain [3], in the asymptotic regime ($t \to +\infty$, $|A| \to +\infty$), (4) narrows down to:

$$P(A) \simeq \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}BA}\mathrm{e}^{-tE_0(B)} \tag{6}$$

where $E_0(B)$ is the ground state energy of H. Moreover, due to large fluctuations of the factor $\mathrm{e}^{\mathrm{i}BA}$, only small B values will give significant contributions to $P(A)$. So, it is enough to compute $E_0(B)$ to lowest order in B by perturbation theory, with the result [3]:

$$E_0(B) = E_0(0) + CB^2 \tag{7}$$

(C is a positive constant depending on the geometry of the system and on the boundary conditions). With (6), it is a simple matter to show that $A/\sqrt{t}$ is a Gaussian variable.

Now, we consider the computation of $P(A)$ when potential (1) is added. In the following, we will stick to the approach developed by Friedberg and Luttinger in [4] where the Lifschitz argument [5] appears in a transparent way.

In 2D and for a zero magnetic field, the average partition function per unit volume reads:

$$Z(B = 0) = \frac{1}{2\pi t}\left\langle \exp\left(-\rho \int \mathrm{d}\vec{r}\,(1 - \mathrm{e}^{-\lambda\int_0^t \delta(\vec{r}(\tau)-\vec{r})\mathrm{d}\tau})\right)\right\rangle_{\{C\}} \tag{8}$$

$\langle\cdots\rangle_{\{C\}}$ stands for an average over all Brownian curves C of length t.

Considering the limit $t \to \infty$, the authors of [4] have shown that:

$$Z(B = 0) = \exp\left(- tE_0(0) - \rho S + \rho \int_D \mathrm{d}\vec{r}\,\mathrm{e}^{-\lambda t\psi_0^2(\vec{r})}\right) \equiv \mathrm{e}^{-tQ} \tag{9}$$

where $E_0(0)$ and ψ_0 are, respectively, the ground state energy and wavefunction for a free particle on a disc D of radius b and area S with Dirichlet boundary conditions. The exponent, Q, in (9) is understood as minimized with respect to b.

From now on, we will drop the subleading ψ_0 term. In those conditions, (9) clearly represents the Lifschitz argument: the low-lying energy states are built in regions free of scatterers.

Now, adding an uniform B field, the only change is a perturbation of E_0 that will, in turn, induce a change in b when Q in (9) is minimized.

Using $E_0(0) = s_1^2/2b^2$, $\psi_0(r) = J_0(s_1 r/b)/\sqrt{N}$ (s_1 is the first zero of the first kind Bessel function J_0), we get:

$$E_0(B) = E_0(0) + CB^2b^2 + \cdots$$

with

$$C = \frac{1}{8Nb^2}\int_0^b r^2 J_0^2\left(\frac{s_1 r}{b}\right) 2\pi r\,\mathrm{d}r = \left(\frac{1}{4s_1^4 J_1^2(s_1)}\right)\int_0^{s_1} J_0^2(x)x^3\,\mathrm{d}x \tag{10}$$

b is obtained by:

$$\frac{\partial}{\partial b}\left(\frac{s_1^2}{2b^2} + CB^2b^2 + \frac{\rho\pi b^2}{t}\right) = 0. \tag{11}$$

Finally, we end up with the results:

$$Z(B) \sim \mathrm{e}^{-\frac{B^2\sigma^2}{2}} \tag{12}$$

$$P(A) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{A^2}{2\sigma^2}} \tag{13}$$

$$\sigma^2 \equiv \langle A^2 \rangle = 2s_1 C t^{3/2}(2\pi\rho)^{-1/2} \approx 0.0523 t^{3/2}\rho^{-1/2}. \tag{14}$$

A is a Gaussian variable that scales like $t^{3/4}$, a situation intermediate between the cases considered above: (i) whole plane allowed ($A \sim t$), (ii) only a bounded domain allowed ($A \sim t^{1/2}$).

Equation (14) does not depend on λ. This is actually the limit $\lambda \to \infty$. First corrections to this expression are of order $\lambda^{-1/2}$ (they come out from the ψ_0 term in (9)). We have checked (14) by numerical simulations on a lattice. The scattering centres being located at the vertices of the lattice, the average over disorder is realized by weighting each N-step random walk C by a factor $\exp(-\rho \sum_i (1 - \mathrm{e}^{-\lambda n_i}))$, n_i being the number of times C has visited site i. When $\lambda \to \infty$, this factor reduces to $\exp(-\rho k)$ with k the number of distinct sites visited by C. The exponent can become very large because k is of order of $N/\ln N$. The width, $\langle A^2 \rangle^{1/2}$, is plotted, in figure 1 (logscale), as a function of N with $\rho = 0.1$. For each simulation point, we have generated 20 000 random walks, N running from 100 to 6000. The straight line is the theoretical calculation equation (14): $\langle A^2 \rangle^{1/2} = 0.407 N^{3/4}$. The agreement is not perfect. Nevertheless, the numerical data clearly exhibit the desired power-law behaviour.

As a remark, let us mention that, in a recent paper [6], Samokhin has computed the winding angle distribution of a Brownian particle wandering in potential (1). Using replica trick and instantons computations, he got, in the limit $t \to \infty$, for the distribution of the winding angle θ around some prescribed point:

$$P(\theta) \sim \frac{1}{1 + x^2} \qquad x = \frac{\theta}{\sqrt{t}} \frac{J_1(s_1)}{\pi Y_0(s_1)} \sqrt{\frac{2}{\pi\rho}}. \tag{15}$$

(The diffusion constant D of [6] has been taken equal to $\frac{1}{2}$, Y_0 is a second kind Bessel function.)

Let us show briefly how this result can be recovered by perturbation theory. $P(\theta)$ is computed in the same way as $P(A)$ except that the uniform magnetic field has to be replaced by a magnetic vortex of strength ϕ located in O. It is well known that the perturbation in ϕ

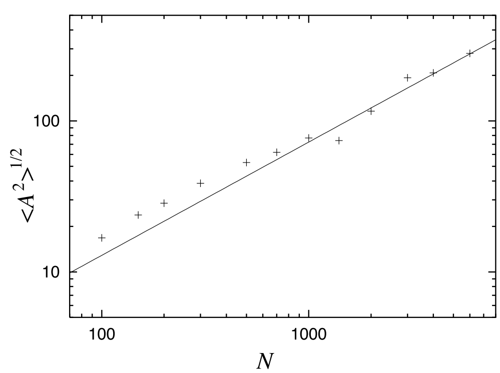

Figure 1. The width of the area distribution, for $\rho = 0.1$, as a function of the length of the random walk (points: numerical simulations; full line: theory). Each point corresponds to 20 000 closed random walks. (For further explanations, see text.)

is singular. Using directly the result of [7] (equation (23), diffusion constant equal to $\frac{1}{2}$), we have:

$$E_0(\phi) = E_0(0) + \frac{|\phi|}{b^2 J_1^2(s_1)} + \cdots. \tag{16}$$

Minimizing the quantity $(E_0(\phi) + \rho\pi b^2/t)$ as before and taking the Fourier transform of $Z(\phi)$, we finally arrive at (15) (with coefficients in complete agreement).

Now, we turn to the Lloyd model [8]. Consider a lattice where each site, j, carries a potential V_j distributed according to a Cauchy law:

$$P_1(V_j) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + V_j^2} \qquad \lambda > 0 \tag{17}$$

with the characteristic function:

$$\langle \mathrm{e}^{\mathrm{i}\mu V_j}\rangle = \mathrm{e}^{-\lambda|\mu|}. \tag{18}$$

Moreover, potentials on different sites are statistically independent.

Now, suppose that a given N-step random walk develops in this environment. It will be weighted by a factor e^{-V} (or $\cosh V$, owing to the fact that P_1, equation(17), is an even function) where V is the 'total potential felt by this curve':

$$V = \sum_i n_i V_i \tag{19}$$

with n_i the number of times this random walk has visited site $i\,(\sum_i n_i = N)$.

Let us compute the probability distribution of the random variable V. From (17) we get:

$$P(X = n_i V_i) = \frac{\lambda}{\pi n_i \left(\lambda^2 + (\frac{X}{n_i})^2\right)} \tag{20}$$

$$\langle \mathrm{e}^{\mathrm{i}\mu X}\rangle = \mathrm{e}^{-\lambda n_i|\mu|}. \tag{21}$$

Statistical independence of the potentials on different sites leads to:

$$\langle \mathrm{e}^{\mathrm{i}\mu V}\rangle = \prod_i \mathrm{e}^{-\lambda n_i|\mu|} = \mathrm{e}^{-\lambda N|\mu|} \tag{22}$$

$$P(V) = \frac{\lambda N}{\pi((\lambda N)^2 + V^2)}. \tag{23}$$

We remark that the probability distribution (23) depends only on the length N of the curve but not on the set of details $\{n_i\}$. Recurrence properties do not play any role in that game and the Lloyd potential will not discriminate curves with different shapes. So, we expect, for the properties of such Brownian curves, the same result as when disorder is absent.

Let us check this fact for $P(A)$. We will use the expression quoted in [9] for $\rho(E)$, the average density of states when a B field is added to the Lloyd potential:

$$\rho(E) = \frac{|B|}{2\pi^2}\sum_{n=0}^{\infty}\frac{\lambda}{(E - E_n)^2 + \lambda^2} \tag{24}$$

with $E_n = (n + \frac{1}{2})|B|$; $\rho(E) \to_{\lambda\to 0} (|B|/2\pi)(\sum_{n\geqslant 0}\delta(E - E_n))$ i.e. the Landau spectrum.

We get, for the partition function (a convergence factor is used):

$$Z(B) = \lim_{\epsilon\to 0^+}\frac{|B|}{2\pi^2}\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-tE}\,\frac{\lambda\mathrm{e}^{-\epsilon(E-E_n)^2}}{(E - E_n)^2 + \lambda^2}\,\mathrm{d}E = \frac{B}{\sinh(\frac{Bt}{2})}F(t,\lambda). \tag{25}$$

$F(t, \lambda)$ depends on disorder but not on the B field. This leads to

$$\frac{Z(B)}{Z(0)} = \frac{\frac{Bt}{2}}{\sinh \frac{Bt}{2}}. \tag{26}$$

Fourier transforming, we are left with (2), i.e. the disorder-free probability distribution $P(A)$.

As a final remark, we could also consider more general probability distributions than (17) for the site potential. For instance, replacing (18) by the characteristic function of Levy's law:

$$\langle \mathrm{e}^{\mathrm{i}\mu V_j} \rangle = \mathrm{e}^{-\lambda|\mu|^\alpha} \qquad 0 < \alpha \leqslant 2 \tag{27}$$

($\alpha = 2$ is Gauss, $\alpha = 1$ is Cauchy, i.e. Lloyd model) we get for V, defined in equation (19):

$$\langle \mathrm{e}^{\mathrm{i}\mu V} \rangle = \mathrm{e}^{-\lambda(\sum_i n_i^\alpha)|\mu|^\alpha}. \tag{28}$$

When $\alpha \neq 1$, this characteristic function depends on the details of the curve through the quantity $Y \equiv \sum_i n_i^\alpha$. The weight of curves with large Y values will be increased. This happens for compact curves when $\alpha > 1$ and for swollen curves when $\alpha < 1$ ($\alpha = 1$ is critical). We are aware that case $\alpha < 1$ deserves further investigation: in particular, we think that it could represent a way to access to properties of self-avoiding random walks [10].

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References

[1] Levy P 1948 *Processus Stochastiques et Mouvement Brownien* (Paris: Editions Jacques Gabay)
[2] Brereton M G and Butler C 1987 *J. Phys. A: Math. Gen.* **20** 3955
Khandekar D C and Wiegel F W 1988 *J. Phys. A: Math. Gen.* **21** L563
Duplantier B 1989 *J. Phys. A: Math. Gen.* **22** 3033
Yor M J 1989 *J. Phys. A: Math. Gen.* **22** 3049
Comtet A, Desbois J and Ouvry S 1991 *J. Phys. A: Math. Gen.* **23** 3563
Antoine M, Comtet A, Desbois J and Ouvry S 1991 *J. Phys. A: Math. Gen.* **24** 2581
[3] Desbois J and Comtet A 1992 *J. Phys. A: Math. Gen.* **25** 3097
[4] Friedberg R and Luttinger J M 1975 *Phys. Rev.* B **12** 4460
[5] Lifschitz I M 1965 *Sov. Phys.–JETP* **7** 549
[6] Samokhin K V 1998 *J. Phys. A: Math. Gen.* **31** 8789
[7] Comtet A, Desbois J and Monthus C 1993 *J. Phys. A: Math. Gen.* **26** 5637
[8] Lloyd P 1969 *J. Phys. C: Solid State Phys.* **2** 1717
Brezin E, Gross D J and Itzykson C 1984 *Nucl. Phys.* B **235** 24
[9] Klein A and Perez J F 1985 *Nucl. Phys.* B **251** 199
[10] Desbois J in preparation